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LETTER TO THE EDITOR

Reunion of vicious walkers: results from ϵ-expansion

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Abstract. The anomalous exponent, η_p, for the decay of the reunion probability of p vicious walkers, each of length N, in d $(= 2-\epsilon)$ dimensions, is shown to come from the multiplicative renormalization constant of a p directed polymer partition function. Using renormalization group (RG) we evaluate η_p to $O(\epsilon^2)$. The survival probability exponent is $\eta_p/2$. For $p = 2$, our RG is exact and η_p stops at $O(\epsilon)$. For $d = 2$, the log corrections are also determined. The number of walkers that are sure to reunite is 2 and has no ϵ expansion.

A question of perennial interest is this [1]: if p vicious (i.e. mutually avoiding) random walkers in d dimensions start from a point in space at time $z = 0$, how does the probability of reunion (i.e., all the p walkers get to the same place) at time $z = N$ decay with N? A complementary question is the decay of the survival probability. Generally, for large N, a power law form is expected signifying certain universality in the behaviour which we explore in this letter.

One of the main motivations to study vicious walkers is to understand and predict the nature of the phase transitions for interface wetting phenomena, melting of the commensurate phase, commensurate–incommensurate (CI) transitions etc, all of which involve walls or string like objects [1, 2]. These walls, which are statistically parallel to the z axis, with fluctuations in the transverse direction may reunite if defects are present in the system. The loops formed thereby are often relevant for criticality [2–6]. They are also important in the context of Abelian sand-pile model of self-organized criticality [7]. These directed strings can be embedded in arbitrary $d+1$ dimensions with d as the transverse spatial dimension. These are called *directed polymers* (DP). Note that the preferred direction of the DP plays the role of time for the random walker.

In spite of its enormous applications, exact results (e.g. the decay exponents) available for reunion of a system of vicious random walkers are restricted to the $d = 1$ dimension only, with an approach for higher dimensions remaining as a challenge [1, 2]. It is clear that the constraints for the walkers to be reunited at one or both the ends and their repulsion do change the exponents drastically. Our focus is on these exponents for $d > 1$.

In this letter, we adopt the DP description. The viciousness becomes an equal time mutual repulsion for the DPs. The reunion and the survival probablities then come from appropriate partition functions for these interacting DPs. Since we are interested in the asymptotic behaviour, i.e. in the large chain length limit, we adopt a continuum model.

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The merit of the path integral technique easily enables us to formulate the model and obtain the exponents of the chain length for the partition functions in arbitrary $d+1$ dimensions through a renormalization group (RG) approach. In this formulation, the dimensionless Hamiltonian for p DPs in $d+1$ dimensions, each of length N, is given by† [8–10]

$$H_p = \frac{1}{2}\sum_{i=1}^{p}\int_0^N \mathrm{d}z\ \dot{\boldsymbol{r}}_i^2 + v_0 \sum_{i>j}\int_0^N \mathrm{d}z\delta^d(\boldsymbol{r}_{ij}(z)) \tag{1}$$

where $\dot{\boldsymbol{r}}_i = \partial \boldsymbol{r}_i/\partial z$, and $\boldsymbol{r}_{ij}(z) = \boldsymbol{r}_i(z) - \boldsymbol{r}_j(z)$, $\boldsymbol{r}_i(z)$ being the d-dimensional coordinate of a point at z on the contour of the ith chain. The first term is the elastic energy, taking care of the connectivity of the chains. The viciousness of the walkers is simulated in the second term by a δ-function interaction at the same z coordinate with $v_0 > 0$. Dimensional analysis shows that v_0 is dimensionless and hence marginal at $d = 2$ (uppercritical dimension). The fact that $d = 2$ is special will be reflected in the later discussion. Our method can be extended to many body interactions—a case to be discussed elsewhere [11].

The reunion probability for p chains follows from the constrained partition function $Z_{R,p}$ for DPs with the restriction that the chains are all tied together at both the ends at origin. A computationally easier quantity is the no-reunion or survival probability. This comes from the partition function, $Z_{S,p}$, with the restriction that the walkers are all tied together at the origin intially, i.e. at $z = 0$, but are free at the other end. These and the exponents are defined generically below ($g = R$ or S) as

$$Z_{g,p} = \int \mathcal{D}\boldsymbol{r}\ F_g\ e^{-H_p}\ \sim N^{-\psi_{g,p}}. \tag{2}$$

Here, $\mathcal{D}\boldsymbol{r}$ stands for the sum over all paths, and F_g denotes the end point constraint, implemented by a $\delta^d(\boldsymbol{r}_i(0))$ for one end of each chain ($g = R, S$) and a $\delta^d(\boldsymbol{r}_i(N) - \boldsymbol{r})$ for the other end of each chain for reunion at $\boldsymbol{r}$. One can consider a more general reunion problem where the walkers can meet anywhere in space at $z = N$. This, $\mathcal{Z}_{R,p}$, requires an extra integration of the partition function of the equation (2) type over the end point coordinate, and $\mathcal{Z}_{R,p} \sim N^{-\Psi_{g,p}}$. We have indeed checked explicitly for a few cases that, the anomalous part of $\psi_{R,p}$ and $\Psi_{R,p}$ are the same. Incidentally, these exponents have recently been used to discuss a novel crossover in the Ising model [6].

Once the exponents are known, one can address the question of the critical number of vicious walkers that are sure to reunite in d dimensions. Conventional analysis (see, e.g., [1]) shows that this number is determined by the convergence criterion of the conditional probability $\mathcal{Z}_{R,p}/Z_{S,p} \sim N^{-\chi}$, $[\chi = \Psi_{R,p} - \psi_{S,p}]$, i.e. whether $\int \mathrm{d}N N^{-\chi}$ is convergent or not. The critical number is then obtained from $\Psi_{R,p} - \psi_{S,p} = 1$. We call this number p_c.

Our approach is to start with a perturbation expansion (diagrammatically) in the coupling constant, and use the renormalization group (RG) approach to go beyond the validity of the perturbation series [9, 10]. The series stumbles on divergences that require standard dimensional regularization to identify the poles. The removal of the divergences demands a renormalization of the interaction parameter and an overall multiplicative renormalization constant [12]. This constant gives rise to an anomalous exponent of the length scale which is unexpected from a conventional dimensional analysis. Such constants were not needed for virial coefficient or other thermodynamic properties of DPs [9, 10] and hence no anomalous dimension ever appeared there.

† z is the preferred direction.

In this letter we first study the two chain reunion problem through an exact RG [13]. It is then extended to the p chain case. We also study in a similar way the no-reunion case, without giving much details. The results then produce a surprising 'super' universality for p_c. Our procedure is general enough to answer many other questions than those studied here. These will be discussed elsewhere.

Had the walkers been non-interacting ($v_0 = 0$), the exponents would follow trivially from the normalized distribution $G(r \mid z) = (2\pi z)^{-d/2} \exp(-r^2/2z)$, for one walk of length z and the end to end distance r. The 'Gaussian' exponents are

$$\psi_{S,p} = 0 \qquad \psi_{R,p} = pd/2 \qquad \text{and} \qquad \Psi_{R,p} = (p-1)d/2. \tag{3}$$

These Gaussian exponents differ from the exact $d = 1$ results for interacting walkers [1,2]

$$\psi_{S,p} = \frac{p(p-1)}{4} \qquad \psi_{R,p} = \frac{p^2}{2} \qquad \text{and} \qquad \Psi_{R,p} = \frac{p^2-1}{2}. \tag{4}$$

The difference between any of these and the corresponding Gaussian exponent is the anomalous dimension to be denoted by $\eta_{S,p}$ (for survival) and η_p (for reunion). A consequence of equations (3) and (4) is that, for $d = 1$, $p_c = 2$ for vicious walkers in contrast to 3 ($1 + 2/d$ for general d) for non-interacting walkers.

One might expect $p_c = 2$ for $d \leqslant 2$ on a simple argument. For two vicious walkers, one can consider the relative walker who starts from one of the nearest neighbours of the origin (NNO) but avoids the origin. Since for $d = 2$ a random walk is recurrent, one might, ignoring the origin avoidance, expect the walker eventually to come to one of the NNOs again, forcing a reunion. Our analysis shows that this is indeed the right answer.

Two chain case. The simplest case that can be calculated exactly is the reunion problem of two walkers. The partition function which restricts the two walkers to be tied at the ends at spatial coordinate $r = 0$ can be calculated with perturbation in the interaction parameter about the free walkers. The diagrams representing the terms of the perturbation series are, e.g., the first three diagrams of figure 1(a) with the left most line omitted. The general term corresponding to DPs with arbitrary number of encounters can be calculated easily using the identity for m Gaussian propagators

$$G^m(r \mid z) = (2\pi z)^{-(m-1)d/2} m^{-d/2} G(r \mid z/m). \tag{5}$$

In terms of the dimensionless coupling constant $u_0 = v_0 L^\epsilon$, ($\epsilon = 2 - d$ and L an arbitrary length scale) the series for the partition function is

$$(2\pi N)^d Z_{R,2} = 1 + \sum_{n=1}^{\infty} (-u_0/4\pi)^n (4\pi N L^{-2})^{n\epsilon/2} \mathcal{G}_{n+1}(\epsilon/2) \tag{6}$$

where $\mathcal{G}_n(x) = \Gamma^n(x)/\Gamma(nx)$ and $\Gamma(x)$ is the standard Gamma function. This series, clearly showing divergences at $\epsilon = 0$, requires renormalization through absorption of these poles. We quote below, for convenience, the formula for renormalized u and the corresponding β-function from [9]

$$u_0 = u[1 - u/(2\pi\epsilon)]^{-1} \qquad \text{and} \qquad \beta(u) = u\epsilon\ [1 - u/(2\pi\epsilon)] \tag{7}$$

with $u = u^* = 2\pi\epsilon$ as the fixed point where $\beta(u) = 0$. This interaction renormalization is not sufficient to absorb even the divergence in the first order term in u_0 of equation (6). Let us define the renormalized partition function as

$$Z_{R,p}|_r = R_{R,p}(u)\ Z_{R,p} \qquad \text{with} \qquad R_{R,p}(u) = 1 + \sum_{n\rangle 0} b_n u^n \tag{8}$$

as the overall multiplicative renormalization constant ($p = 2$). The coefficients b_n are to be chosen to absorb the divergences left over, order by order. Complete removal of such divergences is possible in this case if $b_1 = (\pi\epsilon)^{-1}$ and $b_2 = 3/(4\pi^2\epsilon^2)$. In fact, it is possible to sum the whole series to get $R_{R,2}(u) = [1 - u/(2\pi\epsilon)]^{-2}$.

An appeal to the renormalization group equation [12] immediately shows that $\psi_{R,2}$ is different from the naive engineering dimension ($= d$), equation (3) by $\eta_2 = \gamma_{R,2}(u^*)$ obtained from

$$2\gamma_{R,p}(u) = \beta(u)\frac{\partial}{\partial u}\ln R_{R,p}(u) \tag{9}$$

with $p = 2$. We, therefore, have

$$\eta_2 = \epsilon \qquad \psi_{R,2} = 2 \qquad \text{and} \qquad \Psi_{R,2} = 2 - d/2. \tag{10}$$

in agreement with the $d = 1$ exact results of equation (4) [13].

p chains. Let us first consider the partition function $Z_{R,p}$ for p chains tied together at the origin at both the ends. The terms in the perturbation expansion upto second order in v_0 correspond to the diagrams shown in figure 1(a). The zeroth-order Gaussian term (figure 1(a)(1)) contributes $(2\pi N)^{-pd/2}$.

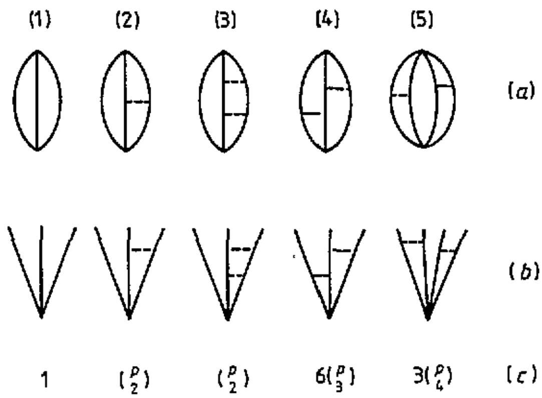

Figure 1. Diagrammatic expansion up to v_0^2 for $p \geqslant 3$ chains tied at (a) both ends and (b) one end. The solid lines represent the chains and the dashed lines represent the interaction at same chain length z (the involved points on the chains have the same spatial coordinates.). Only a few chains are shown. Rules for evaluation: (i) a factor of G for each piece of the solid lines; (ii) $-v_0$ for each dotted line; (iii) an integration over the coordinates of the interaction points and (iv) a time-ordered integration of the z coordinates. See [9, 10]. (c) Symmetry factors. These numbers take into account the number of diagrams with identical contribution but generated either through a mere permutation of the chains or by reversal of time ordering.

The second-order diagrams are of several types: ladder type involving two chains (figure 1(*a*)(3)), three chains connected by interaction (figure 1(*a*)(4)) and two pairs connected independently (figure 1(*a*)(5)). The first and the last of these and the first-order diagram (figure1(*a*)(2)) are already evaluated in equation (6), since these are just two chain terms. The most crucial contribution is from figure 1(*a*)(4) which *connects three chains*. The two chain terms are important for renormalization, but the $O(\epsilon^2)$ contribution to the exponents, ultimately, comes from this diagram.

The contribution of figure 1(*a*)(4) involves $(2\pi N)^{-(p-3)d/2}$ for the non-interacting $(p-3)$ chains. To evaluate the connected piece, one has to use an identity similar to equation (5) for m Gaussian propagators and the Markovian and normalization properties of G. Details will be given elsewhere [11]. The final expression is

$$(2\pi N)^{-pd/2}N^{\epsilon}(4\pi)^{-d}\Big[\mathcal{G}_3(\hat{\epsilon})\,{}_2F_1\left(\hat{\epsilon},\hat{\epsilon};3\hat{\epsilon};\tfrac{3}{4}\right) \\ +\left(\tfrac{3}{4}\right)^{\hat{\epsilon}}\Gamma(-\hat{\epsilon})\Gamma^{-1}(1-\hat{\epsilon})\mathcal{G}_2(\epsilon)\ {}_3F_2\left(\epsilon,\epsilon,1;2\epsilon,1+\hat{\epsilon};\tfrac{3}{4}\right)\Big]$$

where $\hat{\epsilon}=\epsilon/2$, ${}_3F_2$ is the generalized hypergeometric function and $\mathcal{G}_n(x)$ is defined after equation (6). Combining all the terms with appropriate symmetry factors (see figure 1(*c*)) and expanding each one around $\epsilon=0$, we find that

$$\frac{Z_{R,p}}{(2\pi N)^{-pd/2}}=1-\frac{u_0}{2\pi\epsilon}\binom{p}{2}(2+\epsilon\ln x)+\frac{u_0^2}{4\pi^2\epsilon^2}(C-\epsilon D+\epsilon C\ln x)+\ldots, \tag{11}$$

where $C=\binom{p}{2}(p^2-p+1)$, $D=3\binom{p}{3}\ln(3/4)$, and $x=4\pi NL^{-2}$.

Substituting u, from equation (7), for u_0 (note that u_0 still pertains to two-body δ function interaction) in $Z_{R,p}$, equation (11), one sees that the series still requires an overall multiplicative renormalization constant $R_{R,p}(u)=1+b_1u+b_2u^2+\ldots$ (see equation (8)). Minimal subtraction of the poles [12] yields $b_1=\binom{p}{2}/(\pi\epsilon)$ and $b_2=(C+\epsilon D)/(2\pi\epsilon)^2$.

The anomalous exponent comes from $\gamma_{R,p}(u)$ of equation (9). Using the β-function, equation (7), we get

$$2\gamma_{R,p}(u)=\epsilon u[b_1+(2b_2-b_1(2\pi\epsilon)^{-1}-b_1^2)u+\ldots]. \tag{12}$$

Finiteness of $\gamma_{R,p}(u)$ in the limit $\epsilon\to 0$ is guaranteed by the exact cancellation of the $O(\epsilon^{-2})$ term in $2b_2-b_1(2\pi\epsilon)^{-1}-b_1^2$ leaving alone the $O(\epsilon^{-1})$ term. In fact, the finiteness criterion of $\gamma_{R,p}(u)$ as $\epsilon\to 0$ dictates the cancellation of all but the $O(\epsilon^{-1})$ part in all higher orders of u. Therefore, to calculate the $O(\epsilon^2)$ term in $\gamma(u^*)$, no higher-order term in the renormalization constant in $R_{R,p}(u)$ is required. This $O(\epsilon^2)$ term can be traced back to the three chain connected diagram.

We finally obtain from equation (15) the anomalous exponent for reunion as

$$\eta_p\equiv\gamma_{R,p}(u^*)=\binom{p}{2}\,\epsilon+3\binom{p}{3}\,\ln(3/4)\,\epsilon^2+O(\epsilon^3). \tag{13}$$

The survival case, can also be considered in an identical fashion. The diagrams are shown in figure 1(*b*). The unexpected result we find is that, up to $O(u^2)$, the renormalization constant $R_{R,p}(u)$ is the square of the renormalization constant needed for $Z_{S,p}$. We have seen it exactly for $p=2$. Though it seems rather obvious or intuitive [14], still a general proof is lacking. We conclude

$$\psi_{S,p}=\eta_{S,p}=\eta_p/2. \tag{14}$$

This is a scaling relation not hitherto recognized at all.

At the upper critical dimension $d = 2$, the β-function has only the trivial fixed point. Its integration gives $u(L) = u_0[1 + (u_0/2\pi)\ln(L/L_0)]^{-1}$ where $u_0 = u(L_0)$. The renormalization group equation, furthermore, via the method of characteristics [12], leads to the log correction for the reunion probability, as

$$Z_{R,p} \sim N^{-(p-1)} \, [u_0 \ln(N/N_0)]^{-p(p-1)/2} \tag{15}$$

where $N_0 = L_0^2$. Similarly, the survival probability, instead of remaining N-independent as for the Gaussian case, now has a slow decay as $[\ln(N/N_0)]^{-p(p-1)/4}$.

To determine p_c, we now solve $(p-1)d + \eta_p = 2$ in an iterative way. Up to $O(\epsilon^2)$, $p_c = 2$, even after including the log corrections at $d = 2$. Taking into account the exact result [2] for $d = 1$, we predict that $p_c = 2$ for $d < 2$ and is independent of d. Of course, two walkers are not certain to reunite for $d > 2$. Still, the fact that the interaction can make the critical number 'super' universal (at least weakly, i.e. for $d \leqslant 2$), is surprising.

In summary, we emphasize that for the problem of vicious walkers, that can be visualized via the path integrals as directed polymers, an $\epsilon = 2 - d$ expansion is successful in obtaining the essential results. The key feature is the evaluation of the overall multiplicative renormalization constants for the partition functions. The relevant exponents for p chains are computed to $O(\epsilon^2)$. Extensions to still higher orders in ϵ seem not out of reach. Our analysis shows that for $O(\epsilon^2)$, three chains should be collectively aware of each other. Such collective features are expected to continue in higher orders also. For $p = 2$, we obtain exact $O(\epsilon)$ results. A significant result is the scaling relation equation (14) between the anomalous exponents for reunion and survival probabilities. The speciality of the dimension $d = 2$ shows up through the dependence of the probabilities on $\ln N$ with p dependent powers. Furthermore, though 3 (2) non-interacting walkers are sure to unite in one (two) dimensions, we find the number is 2 for vicious walkers, independent of $d < 2$. There seems to be a superuniversality about this number. Numerical verification of these predictions, especially on fractals, would be welcome.

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